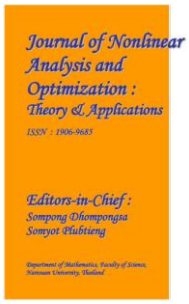

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GENERALIZED CAYLEY INCLUSION PROBLEM INVOLVING XOR OPERATOR IN ORDERED HILBERT

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Abstract

In this paper, we aim to introduce and explore a novel and intriguing problem known as the generalized Cayley inclusion problem, which involves the XOR operation in an ordered positive Hilbert space. We develop a new iterative algorithm with errors to address this system of variational inclusions. Additionally, we examine certain properties of the associated resolvent and Cayley operators using $\oplus$ and $\odot$ operations. Finally, we prove a result on the existence and convergence for the generalized Cayley inclusion problem involving the XOR operation.

Keywords: Variational inclusion; $\oplus$-operation; $\odot$-operation; resolvent operator; Cayley operator; Algorithm;

Convergence.

AMS Subject Classification: 47H05, 47H10, 47J25, 49J40.

1. Introduction

The foundational work on variational inequality was established and explored by Stampacchia [1] and Fichera [2] in the early 1960s. This theory has proven effective in solving problems across various fields such as economics, optimization, elasticity, transportation, and both basic and applied sciences (see [3-12] for more details). Due to its extensive applications, the classical variational inequality problem has been studied and extended in several directions. Among these extensions, variational inclusion holds significant importance.

One of the main challenges in the theory of variational inequality is the development of efficient, practical algorithms. The resolvent operator technique, a generalization of the projection method, has been widely used to address variational inclusion problems. Recent advancements have further refined this technique. Fang and Huang [13] introduced a class of H-monotone operators and expanded the associated class of resolvent operators, building on the work of Ding and Lou [14] and Huang and Fang [15] on maximal monotone operators. For further details, refer to [13,14,16–19] and related references.

The XOR operation ($\oplus$) is a binary operation that acts similarly to addition. It is both associative and commutative, and each element is self-inverse under this operation. In Boolean algebra, XOR corresponds to addition modulo 2. This operation has various practical applications, including data encryption, error detection in digital communication, and parity checking, and it also aids in implementing multi-layer perception in neural networks. In recent years, fixed point theory and its applications have been widely studied in real ordered Banach spaces. Consequently, the study of generalized nonlinear ordered variational inequalities (inclusions) has gained importance. In 2008, Li [20] introduced generalized nonlinear ordered variational inequalities and proposed an algorithm to approximate solutions for a class of these inequalities in real ordered Banach spaces. Since then, several researchers have employed the XOR operation and its variants to solve different classes of variational inequality and inclusion problems in real ordered Hilbert and Banach spaces, see for example [21–32].

Building on this, in this paper, we use the concept of XOR-NODSM mappings, involving the $\oplus$ operation, along with a new resolvent operator technique associated with XOR-NODSM mappings. We introduce and investigate a novel and intriguing problem known as the Generalized Cayley inclusion problem involving XOR operation in ordered positive Hilbert spaces. Additionally, we propose a new iterative algorithm with errors for this system of variational inclusions. Some properties of the associated resolvent operator and Cayley operator are discussed by incorporating $\oplus$ and $\odot$ operations. Finally, we prove the existence and convergence result for the generalized Cayley inclusion problem involving XOR operation.

2. Preliminaries

Throughout this paper, we assume that X is a real ordered positive Hilbert space with the partial ordering "$\leq$" endowed with a norm $\|.\|$ and an inner product $\langle .,. \rangle$, d is the metric induced by the norm $\|.\|$ and 2^{X} is the family of all non-empty subsets of X. First, we recall some known definitions and results which are important to achieve the goal of this paper.

Definition 2.1. A non-empty closed convex subset C of $\|.\|$ is said to be a cone if

(i) For any $x \in C$ and any $\lambda > 0$, $\lambda x \in C$

(ii) For any $x \in C$ and $-x \in C$. Then $x = 0$

(iii) x and y are said to be comparable to each other if and only if either $x \leq y$ or $y \leq x$ and is denoted by $x\ R\ y$.

Definition 2.2. A cone C is called a normal cone if and only if there exists a constant $\lambda_N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq \lambda y, \forall x, y \in X$, where λ_N is called a normal constant of C.

Definition 2.3. For any $x, y \in X,\ x \leq y$ if and only if $y - x \in C$. The relation $\leq$ is a partial ordered relation in X. The real Hilbert Space X endowed with the ordered relation $\leq$ defined by C is called an ordered real Hilbert Space.

Definition 2.4. For arbitrary elements $\mathrm{x}, \mathrm{y} \in \mathrm{X}$, $\mathrm{lub}\{\mathrm{x}, \mathrm{y}\}$ and $\mathrm{glb}\{\mathrm{x}, \mathrm{y}\}$ means least upper bound and greatest lower bound of the set $\{\mathrm{x}, \mathrm{y}\}$. Suppose $\mathrm{lub}\{\mathrm{x}, \mathrm{y}\}$ and $\mathrm{glb}\{\mathrm{x}, \mathrm{y}\}$ exist, some binary operations are defined as follows:

(i) $x \vee y = lub\{x, y\}$

(ii) $x \wedge y = glb\{x, y\}$

(iii) $x \oplus y = (x - y) \vee (y - x)$

(iv) $x \odot y = (x - y) \wedge (y - x)$.

The operations $\vee$, $\wedge$, $\oplus$ and $\odot$ are called *OR*, *AND*, *XOR* and *XNOR* operations, respectively.

Lemma 2.5. If $x\ R\ y$, then lub$\{x, y\}$ and glb$\{x, y\}$ exist, $(x - y)\ R\ (y - x)$ and $0 \leq (x - y) \vee (y - x)$.

Lemma 2.6. For any natural number n, $x\ R\ y_n$ and $y_n\ R\ y^*$ as $n \to \infty$, then $x\ R\ y^*$.

Proposition 2.7. Let $\oplus$ and $\odot$ be XOR and XNOR operations, respectively. Then for all $x, y, u, v \in X;\ \alpha, \beta, \lambda \in R$, the following relations hold:

(i) $x \odot x = 0, x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$

(ii) if $x\ R\ 0$, then $-x \oplus 0 \leq x \leq x \oplus 0$.

(iii) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$.

(iv) $0 \leq (x \oplus y)$ if $x\ R\ y$

(v) if $x\ R\ y$, then $x \oplus y$=0 if and only if $x = y$;

(vi) $(x \oplus y) \odot (u + v) \geq (x \odot u) + (y \odot v)$

(vii) $(x \oplus y) \odot (u + v) \geq (x \odot v) + (y \odot u)$

(viii) If x, y and w are comparative to each other, then $(x \oplus y) \leq (x \oplus w) + (w \oplus y)$

(ix) $\alpha x \oplus \beta x = |\alpha - \beta| x$, if $x\ R\ 0$.

Proposition 2.8. Let K be a normal cone in X with normal constant N, then for each

$x, y \in X$, the following relations hold:

(i) $||0 \oplus 0|| = ||0 \oplus 0|| = 0$;

(ii) $||x \vee y|| \leq ||x|| \vee ||y|| \leq ||x|| + ||y||$.

(iii) $||x \oplus y|| \leq ||x - y|| \leq \lambda_N ||x \oplus y||$;

(iv) If $x\ R\ y$ then $||x \oplus y|| \leq ||x - y||$.

Definition 2.9. Let $A: X \to X$ be a single valued mapping. Then:

(i) A is said to be comparison mapping if for each $x, y \in X,\ x\ R\ y$, then $A(x)\ R\ A(y)$ and $x\ R\ A(x)$ and $y\ R\ A(y)$.

(ii) A is said to be strongly comparison mapping if A is comparison mapping, and $A(x)\ R\ A(y)$ if and only if $x\ R\ y$ for any $x, y \in X$.

Definition 2.10. A mapping $A: X \to X$ is said to be β-ordered compression mapping, if

A is comparison mapping and $A(x) \oplus A(y) \leq \beta(x \oplus y)$, for $0 < \beta < 1$.

Definition 2.11. A single valued mapping $A: X \to X$ is said to be α-Lipschitz-type-continuous if there exists a constant $\alpha > 0$ such that

$$\|A(x) \oplus A(y)\| \leq \alpha\|x - y\|.$$

Definition 2.12. Let $M: X \to 2^X$ be a set-valued mapping, then

(i) M is said to be comparison mapping if for any $v_x \in M(x)$, $x \, R \, v_x$ and if $x \, R \, y$, then for any $v_x \in M(x)$ and any $v_y \in M(y)$, $v_x \, R \, v_y$, $x, y \in X$.

(ii) A comparison mapping M is said to be α-non-ordinary difference mapping, if for each $x, y \in X$, $v_x \in M(x)$ and $v_y \in M(y)$, such that $(v_x \oplus v_y) \oplus \alpha(x \oplus y) = 0$.

(iii) A comparison mapping M is said to be ϑ-ordered rectangular, if there exists a constant $\vartheta > 0$, for any $x, y \in X$, $v_x \in M(x)$ and $v_y \in M(y)$, such that

$$\langle v_x \odot v_y, -(x \oplus y)\rangle \geq \vartheta\|x \oplus y\|^2.$$

(iv) a comparison mapping M is said to be ρ-XOR ordered strongly monotone compression mapping, if for $x \, R \, y$, there exists a constant $\rho > 0$ such that

$$\rho(v_x \oplus v_y) \geq (x \oplus y), \forall x, y \in X, v_x \in M(x), v_y \in M(y).$$

Definition 2.13. Let $A: X \to X$ be a strongly comparison and β-ordered compression mapping. Then a comparison set-valued mapping $M: X \to 2^X$ is said to be (α, ρ)-XOR-NODSM if M is a α-non-ordinary difference mapping and ρ-XOR-ordered strongly monotone mapping and $(A + \rho M)X = X$, for $\alpha, \beta, \rho > 0$.

Definition 2.14. The resolvent operator $J_{\rho,A}^M: X \to X$ is defined by

$$J_{\rho,A}^M(u) = (A + \rho M)^{-1}(u), \forall u \in X. \tag{2.1}$$

Lemma 2.15. Let $A: X \to X$ be a β-ordered compression mapping and $M: X \to 2^X$ be the set-valued ϑ-ordered rectangular mapping with $\vartheta\rho > \beta$. Then the resolvent operator $J_{\rho,A}^M: X \to X$ is single-valued.

Proof. For any given $u \in X$ and a constant $\rho > 0$, let $x, y \in (A + \rho M)^{-1}(u)$, then

$$v_x = \frac{1}{\rho}\big(u \oplus A(x)\big) \in M(x),$$

$$v_y = \frac{1}{\rho}\big(u \oplus A(y)\big) \in M(y).$$

Using (i) and (ii) of Proposition 2.7, we have

$$v_x \odot v_y = \frac{1}{\rho}\big(u \oplus A(x)\big) \odot \frac{1}{\rho}\big(u \oplus A(y)\big)$$

$$= \frac{1}{\rho}\Big(\big(u \oplus A(x)\big) \odot (u \oplus A(y))\Big)$$

$$= -\frac{1}{\rho}\Big(\big(u \oplus A(x)\big) \odot (u \oplus A(y))\Big)$$

$$= -\frac{1}{\rho}\Big((u \oplus u) \oplus \big(A(x) \oplus A(y)\big)\Big)$$

$$= -\frac{1}{\rho}(0 \oplus (A(x) \oplus A(y)))$$

$$\leq -\tfrac{1}{\rho}\big(A(x) \oplus A(y)\big).$$

Thus, we have

$$v_x \odot v_y \leq -\tfrac{1}{\rho}\big(A(x) \oplus A(y)\big) \qquad (2.2)$$

Since, M is ϑ-ordered rectangular mapping, A is β-ordered compression mapping and by using (2.2), we have

$$\vartheta\|x \oplus y\| \leq \langle v_x \odot v_y, -(x \oplus y)\rangle$$

$$\leq \langle -\frac{1}{\rho}\big(A(x) \oplus A(y)\big), -(x \oplus y)\rangle$$

$$\leq \frac{1}{\rho}\langle A(x) \oplus A(y), x \oplus y\rangle$$

$$\leq \frac{1}{\rho}\langle \beta(x \oplus y), x \oplus y\rangle$$

$$= \frac{\beta}{\rho}\langle x \oplus y, x \oplus y\rangle$$

$$= \frac{\beta}{\rho}\|x \oplus y\|^2$$

$$= \frac{\beta}{\rho}\|x \oplus y\|$$

Thus, we have

$$\|x \oplus y\| = 0 \Rightarrow x \oplus y = 0, \text{ for } \vartheta\rho > \beta \qquad (2.2)$$

Therefore, $x = y$. Hence, the resolvent operator is single-valued for $\vartheta\rho > \beta$.

Lemma 2.16. Let $M: X \to 2^X$ be (α, ρ)-XOR-NODSM set-valued mapping with respect to

$J_{\rho,A}^{M}$ and $A: X \to X$ be a strongly comparison mapping with respect to $J_{\rho,A}^{M}$. Then the resolvent operator $J_{\rho,A}^{M}: X \to X$ is a comparison mapping.

Proof. Let $M: X \to 2^X$ be (α,ρ)-XOR-NODSM set-valued mapping with respect to $J_{\rho,A}^{M}$, i.e., M is α-non ordinary difference mapping and ρ-XOR-ordered strongly monotone comparison mapping with respect to $J_{\rho,A}^{M}$ so that $x\, R\, J_{\rho,A}^{M}$.

For any $x, y \in X$, let $x\, R\, y$ and

$$v_{x^*} = \frac{1}{\rho}\left(x \oplus A\left(J_{\rho,A}^{M}(x)\right)\right) \in M\left(J_{\rho,A}^{M}(x)\right), \tag{2.4}$$

$$v_{y^*} = \frac{1}{\rho}\left(y \oplus A\left(J_{\rho,A}^{M}(y)\right)\right) \in M\left(J_{\rho,A}^{M}(y)\right) \tag{2.5}.$$

Since ρ-XOR-ordered strongly monotone, using (2.4) and (2.5), we have

$$x \oplus y \le \rho\left(v_{x^*} \oplus v_{y^*}\right)$$
$$x \oplus y \le \left[(x \oplus A\left(J_{\rho,A}^{M}(x)\right) \oplus (y \oplus A\left(J_{\rho,A}^{M}(y)\right)\right]$$
$$x \oplus y \le (x \oplus y) \oplus \left[A\left(J_{\rho,A}^{M}(x)\right) \oplus A\left(J_{\rho,A}^{M}(y)\right)\right]$$
$$0 \le A\left(J_{\rho,A}^{M}(x)\right) \oplus A\left(J_{\rho,A}^{M}(y)\right)$$
$$0 \le A\left(J_{\rho,A}^{M}(x)\right) \oplus A\left(J_{\rho,A}^{M}(y)\right)$$

That is,

$$0 \le \left[A\left(J_{\rho,A}^{M}(x)\right) - A\left(J_{\rho,A}^{M}(y)\right)\right] \vee \left[A\left(J_{\rho,A}^{M}(y)\right) - A\left(J_{\rho,A}^{M}(x)\right)\right].$$

Hence,

$$0 \le \left[A\left(J_{\rho,A}^{M}(x)\right) - A\left(J_{\rho,A}^{M}(y)\right)\right] or\ 0 \le \left[A\left(J_{\rho,A}^{M}(y)\right) - A\left(J_{\rho,A}^{M}(x)\right)\right].$$

Thus we have

$$A\left(J_{\rho,A}^{M}(x)\right) \ge A\left(J_{\rho,A}^{M}(y)\right), or\ A\left(J_{\rho,A}^{M}(y)\right) \ge A\left(J_{\rho,A}^{M}(x)\right),$$

which implies that $A\left(J_{\rho,A}^{M}(x)\right)\ R\ A\left(J_{\rho,A}^{M}(y)\right)$.

Since A is strongly comparison mapping with respect to $J_{\rho,A}^{M}$. Therefore, $J_{\rho,A}^{M}(x) R J_{\rho,A}^{M}(y)$. Thus, the resolvent operator is a comparison mapping.

Lemma 2.17. Let $M: X \to 2^X$ be (α,ρ)-XOR-NODSM set-valued mapping with respect to $J_{\rho,A}^{M}$ and $A: X \to X$ be a strongly comparison mapping such that

$$A\left(J_{\rho,A}^{M}(x)\right) \oplus\ A\left(J_{\rho,A}^{M}(y)\right)\ R\ J_{\rho,A}^{M}(x) \oplus J_{\rho,A}^{M}(y).$$

Then the following condition holds for $\alpha\rho > \delta, \delta \ge 1$:

$$J_{\rho,A}^{M}(x) \oplus J_{\rho,A}^{M}(y) \le \frac{\delta}{(\alpha\rho \oplus \delta)}(x \oplus y), \forall\, x, y \in X,$$

That is the resolvent operator $J^M_{\rho,A}$ is $\frac{\delta}{(\alpha\rho\oplus\delta)}$-Lipschitz continuous mapping.

Proof. For any $x, y \in X$, noting the fact that M is (α,ρ)-XOR-NODSM set-valued mapping with respect to A and $J^M_{\rho,A}$ and using 2.16, we have

$$\alpha\left[A\left(J^M_{\rho,A}(x)\right) \oplus A\left(J^M_{\rho,A}(y)\right)\right] = v_x \oplus v_y.$$

$$= \frac{1}{\rho}\left((x \oplus y) \oplus \left(A(J^M_{\rho,A}(x)) \oplus A(J^M_{\rho,A}(y))\right)\right)$$

$$\leq \frac{\delta}{\rho}\left[(x \oplus y) \oplus \left(A(J^M_{\rho,A}(x)) \oplus A(J^M_{\rho,A}(y))\right)\right].$$

That is,

$$\frac{\alpha\rho}{\delta}\left(A(J^M_{\rho,A}(x)) \oplus A(J^M_{\rho,A}(x))\right) \leq (x \oplus y) \oplus \left(A(J^M_{\rho,A}(x)) \oplus A(J^M_{\rho,A}(y))\right)$$

Therefore, we have

$$\left(\frac{\alpha\rho}{\delta} \oplus 1\right)\left[A\left(J^M_{\rho,A}(x)\right) \oplus A\left(J^M_{\rho,A}(y)\right)\right] \leq (x \oplus y).$$

Since $A\left(J^M_{\rho,A}(x)\right) \oplus A\left(J^M_{\rho,A}(y)\right) R \left(J^M_{\rho,A}(x) \oplus J^M_{\rho,A}(y)\right)$, we have

$$\left(\frac{\alpha\rho}{\delta} \oplus 1\right)\left[A\left(J^M_{\rho,A}(x)\right) \oplus A\left(J^M_{\rho,A}(y)\right)\right] \leq (x \oplus y)$$

It follows that

$$J^M_{\rho,A}(x) \oplus J^M_{\rho,A}(y) \leq \frac{\delta}{\alpha\rho\oplus\delta}(x \oplus y).$$

That is the resolvent operator $C^M_{\rho,A}$ is $\frac{\delta}{(\alpha\rho\oplus\delta)}$-Lipschitz continuous mapping.

Definition 2.18. The generalized Cayley operator $C^M_{\rho,A}: X \to X$ is defined as

$$C^M_{\rho,A} = \left(2J^M_{\rho,A} - A\right)(x),\ \forall\, x \in X \text{ and } \rho > 0$$

Lemma 2.19. Let $A: X \to X$ be r-Lipschitz continuous mapping. Then the Cayley operator is L-Lipschitz continuous, where $L = \frac{2\delta + r(\alpha\rho+\delta)}{(\alpha\rho\oplus\delta)}$

Proof. For any $x, y \in X$, we have

$$\begin{aligned}\left\|C^M_{\rho,A}(x) \oplus C^M_{\rho,A}(y)\right\| &= \left\|(2J^M_{\rho,A}(x) - A(x)) \oplus (2J^M_{\rho,A}(y) - A(y))\right\| \\ &\leq 2\left\|J^M_{\rho,A}(x) \oplus J^M_{\rho,A}(y)\right\| + \|A(x) \oplus A(y)\| \\ &\leq 2\left(\frac{\delta}{\alpha\rho\oplus\delta}\right)\|x \oplus y\| + r\|x \oplus y\|. \\ &= \left(\frac{2\delta}{\alpha\rho\oplus\delta} + r\right)\|x \oplus y\|.\end{aligned}$$

That is $\left\|C^M_{\rho,A}(x) \oplus C^M_{\rho,A}(y)\right\| \leq L\|x \oplus y\|$, *where* $L = \left(\frac{2\delta}{\alpha\rho\oplus\delta} + r\right)$.

3. Formulation of the Problem

Let X be a real ordered Hilbert Space. Let $N: X \times X \to X$ be a single valued mapping, $M: X \times X \to 2^X, B, T, G: X \to Cc(X)$ be set-valued mappings. Further let $A, g: X \to X$ be mappings such that $R(g) \cap D(M(.,z)) \neq \phi$. We consider the following problem: Find (x, u, v, z), where $x \in X, u \in B(x), v \in T(x), z \in G(x)$ such that

$$\theta \in N(u, v) \oplus C_{\rho,A}^{M(.,z)}(u) \oplus M(g(x), z) \qquad (3.1)$$

We call this problem system of non-linear variational inclusion problems involving $\oplus$ operator (inshort,SNVIP$\oplus$).

Some Special Cases:

1. If $X = H_p, N(u, v) = P(x)$, $C_{\rho,A}^{M(.,z)}(u) = 0, M(g(x), z) = M(f(x), g(x))$. Then, the SNVIP$\oplus$ (3.1) reduces to: Find $x \in H_p, u \in B(x), v \in T(x)$ such that

$$\theta \in N(u, v) \oplus M(f(x), z) \qquad (3.2)$$

Problem 3.2 is studied by Rais Ahmad et al [33].

2. If $X = H_p, N(u, v) = P(x)$, $M(g(x), z) = M(x), C_{\rho,A}^{M(.,z)}(u) = 0$. Then, the SNVIP$\oplus$ (3.1) reduces to: Find $x \in H_p$ such that

$$0 \in P(x) \oplus M(x). \qquad (3.3)$$

Problem 3.3 is studied in [21].

3. If $X = H_p, N(u, v) = 0, M(g(x), z) = M(x)$, $C_{\rho,A}^{M(.,z)}(u) = 0$. Then, the SNVIP$\oplus$ (3.1) reduces to: Find $x \in H_p$ such that

$$0 \in M(x). \qquad (3.4)$$

Problem 3.4 is introduced and studied in [24].

We remark that for suitable choices of the mappings and underlying space, one can get several classes of known and new problems for SNVIP$\oplus$ (3.1).

4. Existence of solution

We give the following lemma which guarantees the existence of solution of SNVIP$\oplus$ (3.1).

Lemma 4.1. Let X be a real ordered positive Hilbert Space. Suppose $A, g: X \to X$ be r-strongly monotone and $M: X \times X \to 2^X$ be A-monotone mapping. Let $N: X \times X \to X$ and $g: X \to X$ be mappings such that $R(g) \cap D\big(M(.,z)\big) \neq \phi$. Let $B, T, G: X \to Cc(X)$ be set-valued mappings. Then SNVIP⊕ (3.1) has the solution (x, u, v, z), where $x \in X$, $u \in B(x)$, $v \in T(x)$, $z \in G(x)$ if and only if it satisfies the following

$$g(x) = J_{\rho,A}^{M(,.z)}\left\{\rho\Big(N(u,v) \oplus C_{\rho,A}^{M(,.z)}(u)\Big) \oplus A\big(g(x)\big) \oplus \rho\theta\right\}.$$

Proof. Suppose

$$g(x) = J_{\rho,A}^{M(,.z)}\left\{\rho\left(N(u,v) \oplus C_{\rho,A}^{M(,.z)}(u)\right) \oplus A\big(g(x)\big) \oplus \rho\theta\right\}$$

$$\Leftrightarrow g(x) = \big(A + \rho M(,.z)\big)^{-1}\left\{\rho\Big(N(u,v) \oplus C_{\rho,A}^{M(,.z)}(u)\Big) \oplus A\big(g(x)\big) \oplus \rho\theta\right\}.$$

$$\Leftrightarrow \big(g(x)\big) + \rho M(g(x), z) \ni \rho\Big(N(u,v) \oplus C_{\rho,A}^{M(,.z)}(u)\Big) \oplus A\big(g(x)\big) \oplus \rho\theta$$

$$\Leftrightarrow \theta \in \Big(N(u,v) \oplus C_{\rho,A}^{M(,.z)}(u)\Big) \oplus A\big(g(x)\big) \oplus M(g(x), z)$$

Based on the above result, we propose the following iterative algorithm for finding the approximate solution of SNVIP⊕ (3.1).

Iterative Algorithm 4.2.

Step 1. For any $\theta \in X$ and $\rho > 0$, choose $x_0 \in X$, $u_0 \in B(x_0)$, $v_0 \in T(x_0)$, and $z_0 \in G(x_0)$ such that $B, T, G: \to 2^X Cc(X)$.

Step 2. Let

$$g(x_{n+1}) = J_{\rho,A}^{M(,.z_n)}\left\{\rho\left(N(u_n, v_n) \oplus C_{\rho,A}^{M(,.z_n)}(u_n)\right) \oplus A\big(g(x_n)\big) \oplus \rho\theta \oplus e_n\right\}.$$

Step 3. Choose $u_{n+1} \in B(x_{n+1})$, $v_{n+1} \in T(x_{n+1})$, $z_{n+1} \in G(x_{n+1})$, such that

$$\|u_{n+1} \oplus u_n\| \le \|u_{n+1} - u_n\| \le (1 + (1+n)^{-1})H\big(B(x_{n+1}), B(x_n)\big),$$
$$\|v_{n+1} \oplus v_n\| \le \|v_{n+1} - v_n\| \le (1 + (1+n)^{-1})H\big(T(x_{n+1}), T(x_n)\big),$$
$$\|z_{n+1} \oplus z_n\| \le \|z_{n+1} - z_n\| \le (1 + (1+n)^{-1})H(G(x_{n+1}), G(x_n).$$

Step 4. Choose error $\{e_n\} \in X$ to take into account the possible inexact computations of the sequences such that, for all $l \in (0,1)$, $\sum_{j=1}^{\infty}\|e_j \oplus e_{j-1}\| \, l^{-j} < \infty$, $\lim_{n\to\infty} e_n = 0$.

Step 5. If $u_{n+1} \in B(x_{n+1})$, $v_{n+1} \in T(x_{n+1})$, $z_{n+1} \in G(x_{n+1})$, satisfy (4.1) to sufficient accuracy, stop: otherwise, $n = n + 1$ and return to Step 2.

Now we give the following result which guarantees the existence of solution of SNVIP⊕ (3.1) and convergence analysis of the sequences generated by the Iterative Algorithm.

Theorem 4.3. Let $K \subset X$ be a normal cone with constant λ_N. Let $N: X \times X \to X$, $C_{\rho,A}^{M(,.z)}: X \to X$, $A, g: X \to X$, $B, T, G: X \to Cc(X)$ and $M: X \times X \to 2^X$ be mappings such that:

(i). N is a_1-Lipschitz-type-continuous in the first argument and a_2-Lipschitz-type-continuous in the second argument.

(ii). $C_{\rho,A}^{M(,.z)}$ is L-Lipschitz-type-continuous.

(iii) A is β-ordered compression mapping.

(iv) g is r-Lipschitz-type continuous and $(g \oplus I)$ is δ-Lipschitz-type continuous.

(v) M is (α, ρ)-XOR-NODSM and θ-ordered rectangular mapping.

(vi) B is γ-H-Lipschitz-type-continuous and T is ν-H-Lipschitz-type-continuous and G isμ-Lipschitz-type-continuous.

If $x_{n+1}\, R\, x_n$, $g(x_{n+1})\, R\, g(x_n)$, for $n = 0,1,2 \ldots$, and the following condition is satisfied:

$$0 < \frac{\lambda_N \rho\delta(a_1\gamma + a_2\nu + L\gamma) + \lambda_N \delta\beta r}{(1-\delta)(\alpha\rho \oplus \delta)} < 1. \tag{4.2}$$

Then SNVIP⊕ (3.1) has a solution (x, u, v, z), where $x \in X, u \in B(x), v \in T(x), z \in G(x)$. Also the sequences generated by the Iterative Algorithm 4.2 converge strongly to x, u, v, z, respectively.

Proof. By Algorithm 4.2 and Proposition 2.7, we have

$$\begin{aligned} 0 &\le g(x_{n+1}) \oplus g(x_n) \\ &= J_{\rho,A}^{M}\left\{\rho\left(N(u_n, v_n) \oplus C_{\rho,A}^{M(,.z)}(u_n)\right) \oplus A(g(\boldsymbol{x_n})) \oplus \rho\theta \oplus e_n\right\} \oplus \\ &\quad J_{\rho,A}^{M}\left\{\rho\left(N(u_{n-1}, v_{n-1}) \oplus C_{\rho,A}^{M(,.z)}(u_{n-1})\right) \oplus A(g(\boldsymbol{x_{n-1}})) \oplus \rho\theta \oplus e_{n-1}\right\} \end{aligned}$$

Now using Proposition 2.8 and Lipschitz-type-continuity of the resolvent operator, we have

$$\begin{aligned} &\|g(x_{n+1}) \oplus g(x_n)\| \\ &\quad \le \lambda_N \left\| J_{\rho,A}^{M}\left\{\rho\left(N(u_n, v_n) \oplus C_{\rho,A}^{M(,.z)}(u_n)\right) \oplus A(g(\boldsymbol{x_n})) \oplus \rho\theta \oplus e_n\right\}\right. \\ &\quad \left. \oplus J_{\rho,A}^{M}\left\{\rho\left(N(u_{n-1}, v_{n-1}) \oplus C_{\rho,A}^{M(,.z)}(u_{n-1})\right) \oplus A(g(\boldsymbol{x_{n-1}})) \oplus \rho\theta \oplus e_{n-1}\right\}\right\| \\ &\le \frac{\lambda_N \delta}{(\alpha\rho \oplus \delta)} \left\| \left\{\rho N(u_n, v_n) \oplus C_{\rho,A}^{M(,.z)}(u_n) \oplus A(g(\boldsymbol{x_n})) \oplus \rho\theta \oplus e_n\right\}\right. \\ &\quad \left. \oplus \left\{\rho N(u_{n-1}, v_{n-1}) \oplus C_{\rho,A}^{M(,.z)}(u_{n-1}) \oplus A(g(\boldsymbol{x_{n-1}})) \oplus \rho\theta \oplus e_{n-1}\right\}\right\| \\ &\quad \le \frac{\lambda_N \delta\rho}{(\alpha\rho \oplus \delta)} \left\| N(u_n, v_n) \oplus C_{\rho,A}^{M(,.z)}(u_n) \oplus N(u_{n-1}, v_{n-1}) \oplus C_{\rho,A}^{M(,.z)}(u_{n-1})\right\| \\ &\quad + \frac{\lambda_N \delta}{(\alpha\rho \oplus \delta)} \|A(g(\boldsymbol{x_n})) \oplus A(g(\boldsymbol{x_{n-1}}))\| + \frac{\lambda_N \delta}{(\alpha\rho \oplus \delta)} \|e_n \oplus e_{n-1}\| \end{aligned}$$

This implies

$$\begin{aligned} \|g(x_{n+1}) \oplus g(x_n)\| &\le \frac{\lambda_N \delta\rho}{(\alpha\rho \oplus \delta)} \|N(u_n, v_n) \oplus N(u_{n-1}, v_{n-1})\| \\ &+ \frac{\lambda_N \delta\rho}{(\alpha\rho \oplus \delta)} \left\| C_{\rho,A}^{M(,.z)}(u_n) \oplus C_{\rho,A}^{M(,.z)}(u_{n-1})\right\| \\ &+ \frac{\lambda_N \delta}{(\alpha\rho \oplus \delta)} \|A(g(\boldsymbol{x_n})) \oplus A(g(\boldsymbol{x_{n-1}}))\| \end{aligned}$$

$$+\frac{\lambda_N\delta}{(\alpha\rho\oplus\delta)}\|e_n\oplus e_{n-1}\| \tag{4.3}$$

Since XOR operator is associative, N is a_1-Lipschitz-type continuous in the first argument and a_2-Lipschitz type continuous in the second argument and B is γ- H -Lipchitz-type-continuous and T is ν-H-Lipschitz-type-continuous, therefore in view of Algorithm 4.2 we have

$$\begin{aligned}
&\|N(u_n, v_n)\oplus N(u_{n-1}, v_{n-1})\|\\
&\quad\le \|N(u_n, v_n)\oplus N(u_{n-1}, v_n)\|\oplus\|N(u_{n-1}, v_n)\oplus N(u_{n-1}, v_{n-1})\|\\
&\quad\le a_1\|u_n\oplus u_{n-1}\| + a_2\|v_n\oplus v_{n-1}\|\\
&\quad\le a_1(1+n)^{-1}H\big(B(x_{n+1}), B(x_n)\big) + a_2(1+n)^{-1}H\big(T(x_{n+1}), T(x_n)\big)\\
&\quad\le a_1\gamma(1+n)^{-1}\|x_n - x_{n-1}\| + a_2\nu\|x_n - x_{n-1}\|
\end{aligned}$$

This implies that

$$\|N(u_n, v_n)\oplus N(u_{n-1}, v_{n-1})\| \le [(a_1\gamma + a_2\nu)(1+n)^{-1}]\|x_n - x_{n-1}\| \tag{4.4}$$

Since A is β-ordered compression mapping, we have

$$\begin{aligned}
\big\|A\big(g(x_n)\big)\oplus A\big(g(x_{n-1})\big)\big\| &\le \beta\|g(x_n)\oplus g(x_{n-1})\|\\
&\le \beta r\|x_n - x_{n-1}\|
\end{aligned} \tag{4.5}$$

Since $C_{\rho,A}^{M(,.z)}$is L-Lipschitz-type continuous, we have

$$\begin{aligned}
\left\|C_{\rho,A}^{M(,.z)}(u_n)\oplus C_{\rho,A}^{M(,.z)}(u_{n-1})\right\| &\le L\|u_n\oplus u_{n-1}\|\\
&\le L\gamma(1+n)^{-1}\|x_n - x_{n-1}\|
\end{aligned} \tag{4.6}$$

Using (4.4)-(4.6) in (4.3), we have

$$\begin{aligned}
&\|g(x_{n+1})\oplus g(x_n)\|\\
&\quad\le \frac{\lambda_N\delta\rho}{(\alpha\rho\oplus\delta)}[(a_1\gamma + a_2\nu)(1+n)^{-1}]\|x_n - x_{n-1}\|\\
&\quad+\frac{\lambda_N\delta\rho}{(\alpha\rho\oplus\delta)}L\gamma(1+n)^{-1}\|x_n - x_{n-1}\| + \frac{\lambda_N\delta}{(\alpha\rho\oplus\delta)}\beta r\|x_n - x_{n-1}\|\\
&\quad+\frac{\lambda_N\delta}{(\alpha\rho\oplus\delta)}\|e_n\oplus e_{n-1}\|\\
&\quad=\left[\frac{\lambda_N\delta\rho[(a_1\gamma+a_2\nu)(1+n)^{-1}]+\lambda_N\delta\rho L\gamma(1+n)^{-1}+\lambda_N\delta\beta r}{\alpha\rho\oplus\delta}\right]\times\\
&\quad\|x_n - x_{n-1}\| + \frac{\lambda_N\delta}{(\alpha\rho\oplus\delta)}\|e_n\oplus e_{n-1}\|
\end{aligned}$$

Hence,

$$\|g(x_{n+1}) \oplus g(x_n)\| \le \frac{\lambda_N \delta\rho(1+n)^{-1}(a_1\gamma + a_2\nu + L\gamma) + \lambda_N\delta\beta r}{\alpha\rho \oplus \delta}\|x_n - x_{n-1}\|$$

$$+ \frac{\lambda_N\delta}{(\alpha\rho \oplus \delta)}\|e_n \oplus e_{n-1}\| \qquad (4.7)$$

Since $(g \oplus I)$ is δ-Lipschitz-type continuous, we have

$$\|x_{n+1} \oplus x_n\| = \|[g(x_{n+1}) \oplus g(x_n)] \oplus [g(x_{n+1}) \oplus x_{n+1} \oplus g(x_n) \oplus x_n]\|$$

$$\le \|[g(x_{n+1}) \oplus g(x_n)] - [g(x_{n+1}) \oplus x_{n+1} \oplus g(x_n) \oplus x_n]\|$$

$$\le \|g(x_{n+1}) \oplus g(x_n)\| + \|(g \oplus I)x_{n+1} (g \oplus I)x_n\|$$

$$\le \frac{\lambda_N \delta\rho(1+n)^{-1}(a_1\gamma + a_2\nu + L\gamma) + \lambda_N\delta\beta r}{\alpha\rho \oplus \delta}\|x_n - x_{n-1}\|$$

$$+ \frac{\lambda_N\delta}{(\alpha\rho \oplus \delta)}\|e_n \oplus e_{n-1}\| + \delta\|x_{n+1} \oplus x_n\|$$

This implies that

$$\|x_{n+1} \oplus x_n\|$$

$$\le \frac{\lambda_N \delta\rho(1+n)^{-1}(a_1\gamma + a_2\nu + L\gamma) + \lambda_N\delta\beta r}{(\alpha\rho \oplus \delta)(1-\delta)}\|x_n - x_{n-1}\|$$

$$+ \frac{\lambda_N\delta}{(\alpha\rho \oplus \delta)(1-\delta)}\|e_n \oplus e_{n-1}\|$$

Since x_{n+1} R x_n, n = 0,1,2, ..., we have

$$\|x_{n+1} - x_n\| \le \Phi_n\|x_n - x_{n-1}\| + \eta\|e_n \oplus e_{n-1}\|, \qquad (4.8)$$

where, $\Phi_n = \frac{\lambda_N \delta\rho(1+n)^{-1}(a_1\gamma + a_2\nu + L\gamma) + \lambda_N\delta\beta r}{(\alpha\rho \oplus \delta)(1-\delta)}$; and $\eta = \frac{\lambda_N\delta}{(\alpha\rho \oplus \delta)(1-\delta)}$

Let $\Phi = \frac{\lambda_N \delta\rho(a_1\gamma + a_2\nu + L\gamma) + \lambda_N\delta\beta r}{(\alpha\rho \oplus \delta)(1-\delta)}$.

It is clear that $\Phi_n \to \Phi$ as n→∞. By(4.2) we know that $0<\Phi<1$ and hence there exists for all $n \ge n_0$ and $\Phi_0 \in (0,1)$ such that $\Phi_n \le \Phi_0$. Therefore, by (4.8), we have

$$\|x_{n+1} - x_n\| \le \Phi_0\|x_n - x_{n-1}\| + \eta\|e_n \oplus e_{n-1}\| \quad (4.9)$$

$$\le \Phi_0^{n-n_0}\|x_{n_0+1} - x_{n_0}\| + \eta\sum_{j=1}^{n-n_0}\Phi_0^{j-1} t_{n-(j-1)}, \qquad (4.10)$$

where $\boldsymbol{t_n} = \|\boldsymbol{e_n} \oplus \boldsymbol{e_{n-1}}\|$, for all $\boldsymbol{n} \ge \boldsymbol{n_0}$. Hence for any $\boldsymbol{m} \ge \boldsymbol{n} \ge \boldsymbol{n_0}$ we have

$$\|\boldsymbol{x_m} - \boldsymbol{x_n}\| \le \sum_{k=n}^{m-1}\|\boldsymbol{x_{k+1}} - \boldsymbol{x_k}\| \le$$

$$\sum_{k=n}^{m-1}\boldsymbol{\Phi_0}^{k-n_0}\|\boldsymbol{x_{n_0+1}} - \boldsymbol{x_{n_0}}\| + \boldsymbol{\eta}\sum_{k=n}^{m-1}\boldsymbol{\Phi_0}^{k}\left[\sum_{k=n}^{m-1}\frac{t_{k-(j-1)}}{\boldsymbol{\Phi_0}^{k-(j-1)}}\right]$$

Since $\sum_{j=1}^{\infty}\|\boldsymbol{e_j} - \boldsymbol{e_{j-1}}\|\boldsymbol{\gamma_j} < \infty$, for all $\mathbf{l_1} \in (\mathbf{0}, \mathbf{1})$ and $0 < \boldsymbol{\Phi_0} < 1$, it follows that $\|\boldsymbol{x_m} - \boldsymbol{x_n}\| \to \mathbf{0}$, as n→∞ and so $\{\boldsymbol{x_n}\}$ is a Cauchy sequence in $\boldsymbol{X}$. Thus there exists $x \in \boldsymbol{X}$ such that $\boldsymbol{x_n} \to \boldsymbol{x}$ as $n \to \infty$. By Algorithm $(\mathbf{4.2})$ and $\boldsymbol{H}$ -Lipschitz continuity of the mappings $\boldsymbol{B}, \boldsymbol{T}, \boldsymbol{G}$, we have

$$\|u_{n+1} \oplus u_n\| \le \|u_{n+1} - u_n\| \le (1 + (1+n)^{-1})\gamma\|x_{n+1} - x_n\|;$$

$$\|v_{n+1} \oplus v_n\| \leq \|v_{n+1} - v_n\| \leq (1 + (1+n)^{-1})\nu\|x_{n+1} - x_n\|;$$
$$\|z_{n+1} \oplus z_n\| \leq \|z_{n+1} - z_n\| \leq (1 + (1+n)^{-1})\mu\|x_{n+1} - x_n\|$$

This shows that $\mathbf{u_n}, \mathbf{v_n}, \mathbf{z_n}$ are all Cauchy sequences. Thus there exist $\mathbf{u}, \mathbf{v}, \mathbf{z} \in \mathbf{X}$ such that $\mathbf{u_n} \to \mathbf{u}, \mathbf{v_n} \to \mathbf{v}, \mathbf{z_n} \to \mathbf{z}$ as $n \to \infty$. Now, we will show that $u \in \mathbf{B}(\mathbf{x}), v \in \mathbf{T}(\mathbf{x}), z \in \mathbf{G}(\mathbf{x})$. We have

$$\begin{aligned} d(\boldsymbol{u}, \boldsymbol{B}(\boldsymbol{x})) &\leq \|\boldsymbol{u} \oplus \boldsymbol{u_n}\| + d(\boldsymbol{u_n}, \boldsymbol{B}(\boldsymbol{x})) \\ &\leq \|\boldsymbol{u} - \boldsymbol{u_n}\| + d(\boldsymbol{u_n}, \boldsymbol{B}(\boldsymbol{x})) \\ &\leq \|\boldsymbol{u} - \boldsymbol{u_n}\| + \boldsymbol{H}(\boldsymbol{B}(\boldsymbol{x_n}), \boldsymbol{B}(\boldsymbol{x})) \\ &\leq \|\boldsymbol{u} - \boldsymbol{u_n}\| + \boldsymbol{\gamma}\|\boldsymbol{x_n} - \boldsymbol{x}\| \to \mathbf{0} \text{ as } n \to \infty. \end{aligned}$$

Since $\boldsymbol{B}(\boldsymbol{x})$ is compact, we have $u \in \boldsymbol{B}(\boldsymbol{x})$ Similarly, we can prove *that* $v \in \boldsymbol{T}(\boldsymbol{x}), z \in \boldsymbol{G}(\boldsymbol{x})$. Thus, we conclude that $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{z})$ such that $x \in \boldsymbol{X}$, $u \in \boldsymbol{B}(\boldsymbol{x})$, $v \in \boldsymbol{T}(\boldsymbol{x}), z \in \boldsymbol{G}(\boldsymbol{x})$ is the solution of SNVIP$\oplus$. This completes the proof.

Remark: Using the technique presented in this paper, one can extend, generalize and unify the results considered by various researchers in this direction. The Algorithm 4.1 is more general than the ones considered in [21-24,26-33] and the related references cited therein.

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